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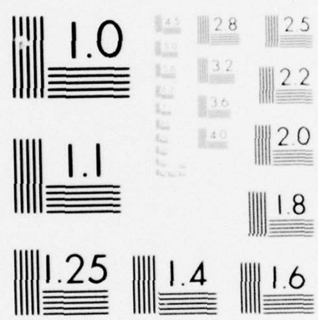

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A LIAPUNOV FUNCTIONAL FOR A MATRIX DIFFERENCE-DIFFERENTIAL EQUATION

21 June 1977

by

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A quadratic positive definite functional that yields necessary and sufficient conditions for the asymptotic stability of the solutions of the matrix difference-differential equation $dx(t)/dt = Ax(t) + Bx(t-\tau)$ is constructed and its structure analyzed. This functional, a Liapunov functional, provides the best possible estimate for the rates of growth or decay of the solutions of this equation. The functional obtained, and its method of construction, are natural generalizations of the same problem for ordinary differential equations, and this relationship is emphasized. An example illustrates the applicability of the results obtained.

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A LIAPUNOV FUNCTIONAL FOR A

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Abstract: A quadratic positive definite functional that yields necessary and sufficient conditions for the asymptotic stability of the solutions of the matrix difference-differential equation $\dot{x}(t) = Ax(t) + Bx(t-\tau)$ is constructed and its structure analyzed. This functional, a Liapunov functional, provides the best possible estimate for the rates of growth or decay of the solutions of this equation. The functional obtained, and its method of construction, are natural generalizations of the same problem for ordinary differential equations, and this relationship is emphasized. An example illustrates the applicability of the results obtained.

1. Introduction.

In this paper we construct a functional for the determination of the asymptotic behavior of the solutions of the linear autonomous matrix difference-differential equation with one delay

$$\dot{x}(t) = Ax(t) + Bx(t-\tau), \quad t > 0,$$

where $x(t)$ is an n-vector function of time, A and B are constant $n \times n$ matrices and $\tau \geq 0$.

In a recent paper [7] such a functional was constructed for the special case when $x(t)$ is a scalar function; there, the Liapunov functional was obtained as the limit, in an appropriate sense, of a Liapunov function constructed by well-known methods for a difference equation approximation of the original functional equation. In this paper, we extend the results in [7] so that they are applicable to the general matrix case; we show that the general form of the functional in the matrix case is a natural generalization of the one obtained in [7] and that, with appropriate modifications, all of the arguments presented in that paper are applicable here. Essential to this generalization was the study, presented in [2], of the existence, uniqueness and structure of the solutions of a special functional differential equation which is intimately connected with the construction of the Liapunov functional presented here.

The Liapunov functionals obtained, and their method of construction, are very natural extensions of the methods used for the construction of Liapunov functions for linear systems of ordinary differential equations. We attempt to bring this relationship into evidence.

A simple example illustrates the applicability of the results obtained, by giving a detailed construction of a functional for a 2×2 system.

The problem of the construction of Liapunov functionals for equations of this type has been previously considered by Repin [8], Datko [3] and Hale [5,6]. This study is in the spirit of the previous ones, but is more specific. The functionals obtained here are more general than the ones used by Hale, and they yield not only sufficient but necessary conditions as well for asymptotic stability. The paper of Repin does not seem to us to be correct, through a simple mistake with profound consequences. Datko [3] has not pursued the problem of construction to the extent presented here, but his ideas have had a deep effect on our approach.

2. The Difference-Differential Equation.

Denote by $L_2([a,b],\mathscr{R}^n)$ the space of Lebesgue square integrable functions defined on $[a,b]$ with values in $\mathscr{R}^n$, and for a fixed $\tau \geq 0$ consider the Hilbert space $\mathscr{A} = \mathscr{R}^n \times L_2([-\tau,0],\mathscr{R}^n)$ with inner product $\langle u_1,u_2\rangle = v_1^T v_2 + \int_{-\tau}^{0} \phi_1^T(\theta)\phi_2(\theta)d\theta$, where $u_i = (v_i,\phi_i) \in \mathscr{A}$, and the naturally induced norm $||(v,\phi)||_{\mathscr{A}}^2 = v^T v + \int_{-\tau}^{0} \phi^T(\theta)\phi(\theta)d\theta$. With $x: [-\tau,\infty) \to \mathscr{R}^n$, for $t \geq 0$ we denote by x_t the function $x_t: [-\tau,0] \to \mathscr{R}^n$, where $x_t(\theta) = x(t+\theta)$.

Consider the matrix difference-differential equation

$$\dot{x}(t) = Ax(t) + Bx(t-\tau), \quad t > 0, \tag{2.1}$$

where A,B are $n \times n$ matrices, $x(t)$ is an n-vector and $\tau \geq 0$, together with the initial conditions

$$x_0(0) = \xi, \quad x_0 = \phi, \tag{2.2}$$

where $(\xi,\phi) \in \mathscr{A}$.

A solution of this initial value problem is a function $x \in L_2([-\tau,t],\mathscr{R}^n)$ for each $t > 0$, such that x is absolutely continuous for $t \geq 0$, it satisfies (2.1) a.e. on $[0,t]$ and $x(0) = \xi$, $x(\theta) = \phi(\theta)$ a.e. for $\theta \in [-\tau,0]$. It is known [1, 6] that (2.1)-(2.2) has a unique solution, defined on $[-\tau,\infty)$, which depends continuously on the initial data in the norm of $\mathscr{A}$.

The initial value problem (2.1)-(2.2) can be rewritten as

$$\frac{d}{dt}\begin{pmatrix} x_t(0) \\ x_t \end{pmatrix} = \mathscr{A}\begin{pmatrix} x_t(0) \\ x_t \end{pmatrix}, \tag{2.3}$$

$$(x_0(0), x_0) = (\xi, \phi) \in \mathscr{M}, \tag{2.4}$$

where

$$\mathscr{A}\begin{pmatrix} x_t(0) \\ x_t \end{pmatrix} = \begin{pmatrix} Ax_t(0) + Bx_t(-\tau) \\ \dfrac{\partial x_t(\theta)}{\partial\theta}, -\tau \leq \theta \leq 0 \end{pmatrix}, \tag{2.5}$$

and this operator has a domain, dense in $\mathscr{M}$, defined by

$$\mathscr{D}(\mathscr{A}) = \{(\xi,\phi) \in \mathscr{M} \mid \phi \text{ is A.C. in } [-\tau,0],$$

$$\phi' \in L_2[-\tau,0],\ \phi(0) = \xi\}.$$

The operator $\mathscr{A}$ is the generator of the C_0-semigroup $\mathscr{S}(t)$, where $\mathscr{S}(t)\colon \mathscr{M} \to \mathscr{M}$ is given by $\mathscr{S}(t)(\xi,\phi) = (x(t), x_t)$, $(x(t), x_t)$ the solution pair of (2.3), (2.4).

It is known [1,6] that there is a constant γ such that the spectrum of $\mathscr{A}$ lies in the left-half plane $\mathrm{Re}(\lambda) \leq \gamma$, and that for every $\in > 0$ there exist a constant $K \geq 1$ such that

$$||\mathscr{S}(t)||_{(\mathscr{M},\mathscr{M})} \leq Ke^{(\gamma+\in)t}. \tag{2.6}$$

The spectrum of $\mathscr{A}$ consists of those complex λ that satisfy the characteristic equation

$$\det[\lambda I - A - Be^{-\lambda\tau}] = 0. \tag{2.7}$$

Finally, [5,6], a useful representation of the solutions of (2.1) is given for every $t, T \geq 0$, by

$$x_{t+T}(0) = S(T)x_t(0) + \int_{-\tau}^{0} S(T-\alpha-\tau)Bx_t(\alpha)d\alpha, \tag{2.8}$$

where the matrix S is the solution of the matrix initial value problem

$$\begin{aligned} &\frac{d}{dt} S(t) = S(t)A + S(t-\tau)B, \\ &S(0) = I,\ S(t) = 0 \quad \text{for} \quad t < 0. \end{aligned} \tag{2.9}$$

We now turn to the construction of a Liapunov functional.

3. A Quadratic Functional.

Associated with the functional differential equation (2.1), or (2.3)-(2.4), and motivated by the results in [7], we wish to consider the real symmetric quadratic form on $\mathscr{A}$, where $(\xi,\phi) \in \mathscr{A}$,

$$\begin{aligned} V(\xi,\phi) = {} & \xi^T M\xi + e^{\delta\tau} \int_{-\tau}^{0} \phi^T(\theta) R e^{2\delta\theta} \phi(\theta) d\theta + \\ & + \xi^T Q(0)\xi + 2\xi^T \int_{-\tau}^{0} Q(\alpha+\tau) e^{\delta(\alpha+\tau)} B\phi(\alpha) d\alpha \\ & + 2\int_{-\tau}^{0}\int_{\alpha}^{0} \phi^T(\alpha) B^T Q(\beta-\alpha) e^{\delta(\alpha+\beta+2\tau)} B\phi(\beta) d\beta d\alpha, \end{aligned} \tag{3.1}$$

where δ is a real number, M,R are constant $n \times n$ real positive definite matrices and $Q(\alpha)$ is a continuously differentiable matrix that is assumed to satisfy the initial value problem for the functional differential equation

$$Q'(\alpha) = (A^T+\delta I)Q(\alpha) + e^{\delta\tau}B^TQ^T(\tau-\alpha), \quad 0 \leq \alpha \leq \tau \tag{3.2}$$

$$Q(0) = Q(0)^T = Q_0, \tag{3.3}$$

where the superscript T denotes transpose and Q_0 is a symmetric but otherwise arbitrary matrix.

Evaluation of the Liapunov functional (3.1) along the solutions of (2.3)-(2.4) yields a function of time, which we denote by $V(t) = V(x_t(0),x_t)$. This function of time is differentiable along the solutions, and a laborious but straightforward computation, which makes use of (3.2) and (3.3), shows that the derivative of this function along the solutions is given by

$$\begin{aligned}\dot{V}(t) = \frac{d}{dt} V(x_t(0),x_t) &= -2\delta V(x_t(0),x_t) + \\ &+ x_t^T(0)[(A^T+\delta I)Q(0) + Q(0)(A+\delta I) + (A^T+\delta I)M + M(A+\delta I) + \\ &\qquad + e^{\delta\tau}B^TQ^T(\tau) + e^{\delta\tau}Q(\tau)B + 2e^{\delta\tau}R]x_t(0) \\ &- e^{\delta\tau}h(x_t(0),x_t(\tau)),\end{aligned} \tag{3.4}$$

where

$$h(x_t(0),x_t(-\tau)) = [x_t^T(0),-e^{-\delta\tau}x_t^T(-\tau)]\begin{bmatrix} R & MB \\ B^TM & R \end{bmatrix}\begin{bmatrix} x_t(0) \\ -e^{-\delta\tau}x_t(-\tau) \end{bmatrix}. \tag{3.5}$$

It is our purpose to show that, through the functional (3.1) and its derivative (3.4) it is possible to estimate the asymptotic behavior of the rate or growth on decay of the solutions for our original functional differential equation (2.3)-(2.4). For this purpose, appropriate choices must be made for the positive definite matrices M and R and of the constant δ, and of the matrix $Q(\alpha)$ determined by (3.2) and (3.3).

To be more specific, denoting by $\gamma = \max\{\mathrm{Re}\,\lambda \mid \det[\lambda I - A - Be^{-\lambda\tau}] = 0\}$, we wish to show that for every $\epsilon > 0$ and $-\delta = \gamma + 2\epsilon$ it is possible to choose matrices M, R and $Q(\alpha)$ satisfying (3.2) and (3.3) so that there exist positive constants c_1, c_2 such that

$$c_1 ||(\xi,\phi)||^2_{\mathscr{A}} \leq V(\xi,\phi) \leq c_2 ||(\xi,\phi)||^2_{\mathscr{A}} , \tag{3.6}$$

and

$$\dot{V}(\xi,\phi) \leq -2\delta V(\xi,\phi). \tag{3.7}$$

These last relationships imply that $\{V(\xi,\phi)\}^{1/2} = |\tilde{|} u(\xi,\phi) |\tilde{|}$ is a norm equivalent to the original norm on $\mathscr{A}$ and that, in this norm

$$|\tilde{|} (x_t(0), x_t) |\tilde{|}_{\mathscr{A}} \leq |\tilde{|} x_0(0), x_0 |\tilde{|}_{\mathscr{A}} e^{-\delta\tau}, \tag{3.8}$$

whereas, in the original norm on $\mathscr{A}$,

$$||(x_t(0), x_t)||_{\mathscr{A}} \leq \left(\frac{c_2}{c_1}\right)^{1/2} ||(x_0(0), x_0)||_{\mathscr{A}} e^{-\delta\tau}. \tag{3.9}$$

These estimates are precisely those stated in (2.6) and are the best possible ones. It should be noted that the norm induced by the square root of the Liapunov functional is the best possible one in the sense that it yields (2.6) with $K = 1$. Moreover, if $\gamma < 0$, the above remarks show that the Liapunov functional (3.1) yields uniform exponential asymptotic stability.

Consider, first of all, the appropriate choice for the matrix $Q(\alpha)$, a solution of (3.2), (3.3) for $0 \le \alpha \le \tau$. In [2] it was shown that equation (3.2) with initial conditions (3.3) has a unique solution; moreover, it was shown that the linear vector space of all solutions of (3.2) has dimension n^2. Indeed, introducing the notation

$$Q(\alpha) = (q_{ij}(\alpha)) = \begin{bmatrix} q_{1*}(\alpha) \\ \vdots \\ q_{n*}(\alpha) \end{bmatrix} = [q_{*1}(\alpha), \ldots, q_{*n}(\alpha)],$$

where $q_{i*}(\alpha)$ and $q_{*j}(\alpha)$ are, respectively, the i^{th} row and the j^{th} column of $Q(\alpha)$, defining the n^2-vector

$$q(\alpha) = (q_{1*}(\alpha), \ldots, q_{n*}(\alpha))^T,$$

and defining the matrix $R(\alpha) = Q^T(\tau-\alpha)$, equation (3.2) reduces, through the use of the Kronecker (or direct) product of two matrices, to the $2n^2$ system of ordinary differential equations

$$\frac{d}{d\alpha}\begin{bmatrix} q(\alpha) \\ r(\alpha) \end{bmatrix} = \begin{bmatrix} (A+\delta I)^T \otimes I & e^{\delta\tau}B^T \otimes I \\ -I \otimes e^{\delta\tau}B^T & -I \otimes (A+\delta I)^T \end{bmatrix}\begin{bmatrix} q(\alpha) \\ r(\alpha) \end{bmatrix}, \qquad (3.10)$$

with the auxiliary condition

$$q(\tfrac{\tau}{2}) = [f_{1*},\ldots,f_{n*}]^T,\ r(\tfrac{\tau}{2}) = [f_{*1}^T,\ldots,f_{*n}^T]^T, \tag{3.11}$$

where F is an arbitrary $n \times n$ matrix. The solution of (3.10), subject to (3.11), yields solutions $Q(\alpha)$ with n^2 parameters that can always be chosen to satisfy any arbitrary condition of the type $Q(0) = Q(0)^T = Q_0$. In [2] the structure of these solutions and simple methods of computation that take advantage of the particular structure of the equation are presented.

Associated with equation (3.2)-(3.3) is an integral, whose structure is motivated by a similar integral for ordinary differential equation. Indeed, let W be a symmetric matrix and let $S(t)$ be the solution of equation (2.9). Consider the expression

$$\tilde{Q}(\alpha) = \int_0^\infty S^T(u)e^{\delta u}WS(u-\alpha)e^{\delta(u-\alpha)}du. \tag{3.12}$$

Since, for every $\epsilon > 0$, $||S(t)|| \leq \tilde{K}e^{(\gamma+\epsilon)t}$ for some $\tilde{K} \geq 1$ and since $\delta = -\gamma - 2\epsilon$, it follows that this integral converges. Moreover, it immediately follows from the definition of $\tilde{Q}(\alpha)$ that $\tilde{Q}(\alpha) = \tilde{Q}^T(-\alpha)$, $\tilde{Q}(0) = \tilde{Q}^T(0)$ and, since $S(t)$ satisfies (2.9), that

$$\tilde{Q}'(\alpha) = -\tilde{Q}(\alpha)(A+\delta I) - \tilde{Q}(\alpha+\tau)e^{\delta\tau}B - S^T(\alpha)e^{\delta\alpha}W.$$

Since $\tilde{Q}'(\alpha) = \frac{d}{d\alpha}[\tilde{Q}^T(-\alpha)]$ for $\alpha \neq 0$, we also obtain that

$$\tilde{Q}'(\alpha) = (A+\delta I)^T\tilde{Q}(\alpha) + B^Te^{\delta\tau}\tilde{Q}^T(\tau-\alpha) + S^T(-\alpha)e^{\delta\alpha}W.$$

From these relationships it follows, since, for $\alpha > 0$, $S(-\alpha) = 0$, that $\tilde{Q}(\alpha)$ satisfies

$$\tilde{Q}'(\alpha) = (A^T+\delta I)\tilde{Q}(\alpha) + B^T e^{\delta\tau} Q^T(\tau-\alpha), \quad 0 \le \alpha \le \tau$$
$$\tilde{Q}^T(0) = \tilde{Q}(0) = \int_0^\infty S^T(u)e^{\delta u} WS(u)e^{\delta u}du \tag{3.13}$$

and, moreover, given the continuity of $Q(\alpha)$, that

$$\tilde{Q}'(0) + \tilde{Q}'^T(0) = (A^T+\delta I)\tilde{Q}(0) + \tilde{Q}(0)(A+\delta I) + B^T e^{\delta T} Q^T(\tau) + Q(\tau)e^{\delta\tau}B$$
$$= -S^T(0)W = -W. \tag{3.14}$$

These observations, and the uniqueness of the solutions of (3.2)-(3.3), show that $\tilde{Q}(\alpha)$ as defined by (3.12) is the unique solution of (3.2)-(3.3) with the initial conditions prescribed by the second equation of (3.13).

It is transparent that to each constant symmetric positive definite matrix W there corresponds a $\tilde{Q}(\alpha)$ given by (3.12) and that this is the unique solution of (3.2)-(3.3) with initial condition $\tilde{Q}(0) = \tilde{Q}(0)^T = \int_0^\infty S^T(u)e^{\delta u}WS(u)e^{\delta u}du$ which is positive definite. Conversely, to each $Q(0) = Q^T(0) = Q_0$, the unique solution of (3.2)-(3.3) yields a differentiable matrix function $Q(\alpha)$ that, through (3.14) defines a unique symmetric matrix W so that, for this W, (3.12) yields the unique solution to (3.2)-(3.3) with the prescribed initial conditions. These remarks show that the map $W \to \tilde{Q}(0)$, defined by

$$\tilde{Q}(0) = \int_0^\infty S^T(u)e^{\delta u}WS(u)e^{\delta u}du,$$

as a map on the space of $n \times n$ symmetric matrices is one-to-one, onto, and it maps positive definite matrices W into positive definite matrices $\tilde{Q}(0)$.

With this particular characterization of the matrix function $\tilde{Q}(\alpha)$ it is possible to bring into evidence the particular structure of the Liapunov functional (3.1). Indeed, substitution of (3.12) into (3.1) for $Q(\alpha)$ yields, after some rearrangements and interchanges of integrals, that

$$\begin{aligned} V(\xi,\phi) = {} & \xi^T M\xi + e^{\delta\tau}\int_{-\tau}^{0} \phi^T(\theta) R e^{2\delta\theta}\phi(\theta)d\theta \\ & + \int_0^\infty \{S(u)\xi + \int_{-\varepsilon}^{0} S(u-\alpha-\tau)B\phi(\alpha)d\alpha\}^T We^{2\delta u}\{S(u)\xi \\ & + \int_{-\tau}^{0} S(u-\alpha-\tau)B\phi(\alpha)d\alpha\}du, \end{aligned} \tag{3.15}$$

or, evaluating this functional on the solutions of our equation, and using (2.8),

$$\begin{aligned} V(x_t(0),x_t) = {} & x_t^T(0)Mx_t(0) + e^{\delta\tau}\int_{-\tau}^{0} x_t^T(\theta)Re^{2\delta\theta}x_t(\theta)d\theta \\ & + \int_0^\infty x_{t+u}(0)We^{2\delta u}x_{t+u}(0)du. \end{aligned} \tag{3.16}$$

Similarly, in this notation, equation (3.4) becomes

$$\begin{aligned} \dot{V}(x_t(0),x_t) = {} & -2\delta V(x_t(0),x_t) + x_t^T(0)[-W + (A^T+\delta I)M + M(A+\delta I) + \\ & + 2e^{\delta\tau}R]x_t(0) + [x_t^T(0), -e^{-\delta\tau}x_t^T(-\tau)]\begin{bmatrix} R & MB \\ B^T M & R \end{bmatrix}\begin{bmatrix} x_t(0) \\ -e^{-\delta\tau}x_t^T(-\tau) \end{bmatrix}. \end{aligned} \tag{3.17}$$

Given the nonnegative nature of the last term in (3.16) it follows that

$$\min(\lambda_{min}(M), e^{-|\delta|\tau}\lambda_{min}(R))||(\xi,\phi)||^2_{\mathscr{A}} \leq V(\xi,\phi),$$

yielding a value of c_1, for equation (3.6), given by

$$c_1 = \min(\lambda_{min}(M), e^{-|\delta|\tau}\lambda_{min}(R)). \tag{3.18}$$

Similarly, since, from (2.6),

$$||x_u(0)||_{\mathscr{R}^n} \leq ||(x_u(0), x_u)||_{\mathscr{A}} \leq Ke^{(\gamma+\epsilon)u}||(x_0(0), x_0)||_{\mathscr{A}},$$

it follows that

$$V(\xi,\phi) \leq [\max\{\lambda_{max}(M), e^{|\delta|\tau}\lambda_{max}(R)\} + \frac{\lambda_{max}(W)}{2\epsilon} K^2]||(\xi,\phi)||^2_{\mathscr{A}}$$

yielding, for equation (3.6), a value of c_2 given by

$$c_2 = \max[\lambda_{max}(M),\ e^{|\delta|\tau}\lambda_{max}(R)] + \frac{\lambda_{max}(W)}{2\epsilon} K^2. \tag{3.19}$$

It remains to be shown that equation (3.7) holds. For this purpose inspection of equation (3.17) indicates that it is necessary and sufficient to show that it is possible, by appropriate choices of positive definite matrices W, R and M, to have the matrix

$$G = \begin{bmatrix} W - (A+\delta I)^T M - M(A+\delta I) - Re^{\delta\tau} & MBe^{\delta\tau} \\ B^T Me^{\delta\tau} & Re^{\delta\tau} \end{bmatrix} \tag{3.20}$$

positive semidefinite. But this is always possible; indeed, a particularly simple choice of such matrices is to let W, R and M to be nonnegative multiples of the identity matrix, i.e. $M = I$, $R = k_R I$ and $W = k_W I$. With $k_W >> k_R >> 1$, the matrix G is clearly positive definite, hence its determinant is positive, and the matrix will become semidefinite only when the determinant vanishes. But, for the above choice of matrices we have

$$\det(G) = \det(k_R I e^{\delta\tau}) \det[k_W I - (A+A^T) - 2\delta I - e^{\delta\tau}(k_R I + \frac{1}{k_R} B^T B)];$$

Letting $k_R = \sqrt{\lambda_{max}(B^T B)}$ and

$$k_W = \max\{0, \lambda_{max}(A+A^T) + 2\delta + 2e^{\delta\tau}\sqrt{\lambda_{max}(B^T B)}\},$$

then (3.20) is positive semidefinite and therefore, for these choices of M, R and W,

$$\dot{V}(x_t(0), x_t) \leq -2\delta V(x_t(0), x_t). \tag{3.21}$$

It is to be remarked that, with these choices, the Liapunov functional constructed is particularly simple and reduces to

$$V(x_t(0),x_t) = x_t^T(0)x_t(0) + e^{\delta\tau}\sqrt{\lambda_{max}(B^TB)}\int_{-\tau}^{0} x_t^T(\theta)x_t(\theta)e^{2\delta\theta}d\theta +$$

$$+ \max\{0,\lambda_{max}(A+A^T) + 2\delta + 2e^{\delta\tau}\sqrt{\lambda_{max}(B^TB)}\}. \qquad (3.22)$$

$$\cdot \int_0^{\infty} x_{t+s}^T(0)x_{t+s}(0)e^{2\delta s}ds.$$

This functional is a direct generalization of that used in [7], and generalizes those used by Hale [5,6] and Datko [3]. We recapitulate the above results in the form of a

Theorem: Given the retarded equation $\dot{x}(t) = Ax(t) + Bx(t-\tau)$ and the Liapunov functional V given by equation (3.1), if $\gamma = \max\{\text{Re}\,\lambda \mid \det[\lambda I-A-Be^{-\lambda\tau}] = 0\}$ and $\epsilon > 0$, then there exist constant positive definite matrices M and R and a differentiable matrix $Q(\alpha)$, $0 \leq \alpha \leq \tau$ with $Q(0) = Q(0)^T$ such that the functional V is positive definite, bounded above, and $\dot{V} \leq 2(\gamma+\epsilon)V$.

Of course, if $\gamma < 0$, the above result implies exponential asymptotic stability; moreover, the rate of decay is precisely the expected one.

4. A Comparison with Ordinary Differential Equations.

It is our object, in this section, to point out the intimate relationship between the results obtained for our functional equation and the classical results for the construction of Liapunov functions for ordinary differential equations.

Recall, [4], that given the system of ordinary differential equations

$$\dot{x}(t) = Cx(t), \tag{4.1}$$

where C is an $n \times n$ matrix, a Liapunov function for this system can be always taken as the quadratic form

$$V(x) = x^T Px, \tag{4.2}$$

where P is a positive definite matrix. Moreover, if $\gamma = \max\{\mathrm{Re}\,\lambda \mid \det[\lambda I - C] = 0\}$ and if, for $\in > 0$, $-\delta = \gamma + 2\in$, then given an arbitrary positive definite matrix W, the algebraic equation

$$(C+\delta I)^T P + P(C+\delta I) = -W, \tag{4.3}$$

has a unique solution P which is positive definite. This matrix P, if used in (4.2) along the solutions of the differential equation (4.1) yields, upon differentiation, that

$$\dot{V}(x(t)) = -2\delta V(x(t)) - x^T Wx \leq -2\delta V(x(t)). \tag{4.4}$$

Furthermore, the unique positive definite solution P of (4.3) can be obtained as the integral

$$P = \int_0^\infty e^{A^T u} e^{\delta u} W e^{Au} e^{\delta u} du. \tag{4.5}$$

Let us now bring into evidence the relationship between the results obtained in the previous section and these results. The functional differential equation under consideration is

$$\dot{x}(t) = Ax(t) + Bx(t-\tau), \quad t > 0 . \tag{4.1'}$$

Once again, we assume that $\gamma = \max\{\mathrm{Re}\,\lambda \mid \det[\lambda I - A - Be^{-\lambda\tau}\} = 0\}$ and, for $\in > 0$, let $-\delta = \gamma + 2\in$. The Liapunov functional is then of the form

$$V(x_t(0),x_t) = x_t^T(0)Mx_t(0) + e^{\delta\tau}\int_{-\tau}^{0} x_t^T(\theta)Re^{2\delta\theta}x_t(\theta)d\theta \tag{4.2'}$$

$$+ x_t^T(0)Q(0)x_t(0) + 2x_t^T(0)\int_{-\tau}^{0} Q(\alpha+\tau)e^{\delta(\alpha+\tau)}Bx_t(\tau)d\theta +$$

$$+ 2\int_{-\tau}^{0}\int_{\alpha}^{0} x_t^T(\alpha)B^TQ(\beta-\alpha)e^{\delta(\alpha+\beta+2\tau)}Bx_t(\beta)d\beta d\alpha .$$

The choice of the positive definite symmetric matrices M and R in this expression is rather arbitrary. Their purpose is to insure the strict positive definiteness of the functional on the Hilbert space $\mathscr{A}$; it should be noted that if $M = R = 0$, the functional (4.2') is positive, but does not satisfy a relationship of the type $V(x_t(0),x_t) \geq c_1||(x_t(0),x_t)||_{\mathscr{A}}^2$ for $c_1 > 0$. The requirement that the matrix G of equation (3.20) be positive semidefinite is always satisfied for $M = R = 0$ and W positive semidefinite. Given an arbitrary positive definite matrix W, it is always possible to select positive definite matrices M and R so that G is positive semidefinite.

The choice of the continuously differentiable matrix $Q(\alpha)$, $0 \leq \alpha \leq \tau$ is critical. It must satisfy, given an arbitrary positive definite matrix W, the functional equation

$$Q'(\alpha) = (A^T+\delta I)Q(\alpha) + B^T e^{\delta\tau} Q^T(\tau-\alpha), \quad 0 \leq \alpha \leq \tau,$$
$$Q(0) = Q^T(0), \tag{4.3'}$$

with the condition

$$(A^T+\delta I)Q(0) + Q(0)(A+\delta I) + B^T e^{\delta\tau} Q^T(\tau) + Q(\tau)e^{\delta\tau}B = -W. \tag{4.3"}$$

In the previous section it was shown that such a $Q(\alpha)$ always exists and is unique.

With such a choice of $Q(\alpha)$, then one obtains that, along the solutions of the functional differential equation (4.1')

$$\dot{V}(x_t(0),x_t) \leq -2\delta V(x_t(0),x_t). \tag{4.4'}$$

Moreover, such a matrix $Q(\alpha)$ exists, is unique, and a representation of it is given by the integral

$$Q(\alpha) = \int_0^\infty S^T(u)e^{\delta u} WS(u-\alpha)e^{\delta(u-\alpha)}du, \tag{4.5'}$$

where $S(t)$ is the solution of equation (2.9).

The strong relationship between the unprimed and primed equations is now clear. Indeed, note that for $\tau = 0$, the matrix C in (4.1) becomes A + B and the matrix P in (4.3) becomes

$Q(0) + M$ and all the primed equations become the unprimed ones.

Equations (4.3') and (4.3") are of a much more complex nature than the familiar algebraic equation (4.3). However, in spite of its appearance, the linear vector space of the solutions of (4.3') is not infinite dimensional but, as was pointed out in the previous section, has dimension n^2. Hence, although the problem of construction of the Liapunov functional for the functional differential equation does not reduce to the solution of an algebraic equation such as (4.3); it reduces to the solution of a linear differential equation, equation (3.10), which is accomplished without difficulty.

5. An Example.

To illustrate the application of the method developed here, we wish to briefly outline the construction of an appropriate Liapunov functional for a simple two dimensional example. For this purpose, consider the system

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) = \begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix} x(t-\tau), \qquad (5.1)$$

where $\tau = 6$. In the notation of Section 3, the value $\delta = \frac{1}{30}$ is selected as is indicated by a simple examination of the characteristic equation of the system. For the construction of the Liapunov functional (3.1), appropriate positive definite matrices M and R must be selected and a matrix function $Q(\alpha)$ that satisfies (3.2), (3.3) and (3.14) must be determined given a preselected positive definite matrix W.

As indicated in Section 3, an appropriate choice for the matrices M,R and W is given by

$$M = I,$$

$$R = \sqrt{\lambda_{\max}(B^TB)}\ I = \sqrt{20}\ I, \tag{5.2}$$

$$W = \max\{0, \lambda_{\max}(A+A^T) + 2\delta + 2e^{\delta\tau}\sqrt{\lambda_{\max}(B^TB)}\}I = 7{,}246I = k_W I.$$

All that remains is to determine the unique matrix function $Q(\alpha)$ that satisfies (3.2), (3.3) and (3.14). The general form of this matrix function is determined by obtaining the solutions of the differential equation (3.10) following the methods indicated in [2]; in a straightforward although somewhat lengthy computation one obtains that the general form of this matrix

$$Q(\alpha) = \begin{bmatrix} q_{11}(\alpha) & q_{12}(\alpha) \\ q_{21}(\alpha) & q_{22}(\alpha) \end{bmatrix}, \quad 0 \le \alpha \le \tau, \tag{5.3}$$

is of the form

$$q_{11}(\alpha) = \beta_2\left[e^{\sqrt{f(\delta)}(\alpha-\tau/2)} + g_1(\delta)e^{-\sqrt{f(\delta)}(\alpha-\tau/2)}\right],$$

$$q_{12}(\alpha) = \beta_2\left[\eta_{12}(\delta)e^{\sqrt{f(\delta)}(\alpha-\tau/2)} + g_1(\delta)\eta_{21}(\delta)e^{-\sqrt{f(\delta)}(\alpha-\tau/2)}\right] +$$

$$+ \beta_4\, e^{(2-\delta)(\alpha-\tau/2)},$$

$$q_{21}(\alpha) = \beta_1 e^{(-5+\delta)(\alpha-\tau/2)} + \beta_2\left[\eta_{21}(\delta)e^{\sqrt{f(\delta)}(\alpha-\tau/2)} + \right.$$

$$\left. + g_1(\delta)\eta_{12}(\delta)e^{-\sqrt{f(\delta)}(\alpha-\tau/2)}\right]$$

$$+ \beta_4 g_2(\delta)e^{-(2-\delta)(\alpha-\tau/2)}, \tag{5.4}$$

and

$$q_{22}(\alpha) = \beta_2\left[\eta_{22}(\delta)e^{\sqrt{f(\delta)}(\alpha-\tau/2)} + g_1(\delta)\eta_{22}(\delta)e^{-\sqrt{f(\delta)}(\alpha-\tau/2)}\right] +$$

$$+ \beta_3 e^{(\delta-2)(\alpha-\tau/2)} + \beta_4\eta_{11}(\delta)\left[e^{(2-\delta)(\alpha-\tau/2)} + \right.$$

$$\left. + g_1(\delta)e^{-(2-\delta)(\alpha-\tau/2)}\right].$$

In these expressions the indicated functions of δ are, respectively, given by

$$\begin{aligned}
f(\delta) &= (5-\delta)^2 - 16e^{2\delta\tau}, \\
g_1(\delta) &= \frac{\sqrt{f(\delta)} + (5-\delta)}{4e^{\delta\tau}}, \\
g_2(\delta) &= \frac{7-2\delta}{4e^{\delta\tau}}, \\
\eta_{11}(\delta) &= \frac{7-2\delta}{8-4\delta} \\
\eta_{12}(\delta) &= \frac{2e^{\delta\tau}}{[2-\delta-\sqrt{f(\delta)}]g_1(\delta)}, \\
\eta_{21}(\delta) &= \frac{2e^{\delta\tau}g_1(\delta)}{2-\delta+\sqrt{f(\delta)}},
\end{aligned} \tag{5.5}$$

and

$$\eta_{22}(\delta) = \eta_{12}(\delta)\eta_{21}(\delta).$$

The constants $\beta_1, \beta_2, \beta_3$ and β_4 are to be determined from the four equations obtained from the relation $Q(0) = Q(0)^T$ (i.e., $q_{12}(0) = q_{21}(0)$) and from equation (3.14) (i.e., $Q'(0) + Q'(0)^T = -W = -k_W I$).

Solution of these four linear equations for the indicated value of $\delta = \frac{1}{30}$ yields the values

$$\begin{aligned} \beta_1 &= -0.115.10^{-5}k_W, \quad \beta_2 = 0.321.10^{-1}k_W, \\ \beta_3 &= 0.242.10^{-3}k_W, \quad \beta_4 = -0.622.10^{-3}k_W. \end{aligned} \tag{5.6}$$

With these values, it is easily computed that $Q(0) = Q(0)^T$, and that this matrix is positive definite.

REFERENCES

[1] Borisovic, J.G. and A.S. Turbabin, On the Cauchy Problem for Linear Nonhomogeneous Differential Equations with Retarded Argument, Soviet Math. Doklady 10, 1969, pp. 401-405.

[2] Castelan, W.B. and E.F. Infante, On a Functional Equation Arising in the Stability Theory of Difference-Differential Equations, to appear, Quarterly of Applied Mathematics.

[3] Datko, R., An Algorithm for Computing Liapunov Functionals for some Differential-Difference Equations, Ordinary Differential Equations, 1971 NRL-MRC Conference, Academic Press, 1972, p. 387-398.

[4] Hale, J.K., Ordinary Differential Equations, Wiley-Interscience, 1967.

[5] Hale, J.K., Functional Differential Equations, Appl. Math. Science Series, Springer-Verlag, 1971.

[6] Hale, J.K., Theory of Functional Differential Equations, Springer-Verlag, 1977.

[7] Infante, E.F. and J.A. Walker, A Liapunov Functional for a Scalar Differential-Difference Equation, submitted for publication.

[8] Repin, I.M., Quadratic Liapunov Functionals for Systems with Delays, Prikl. Mat. Mech. 29, 1965, pp. 564-566.